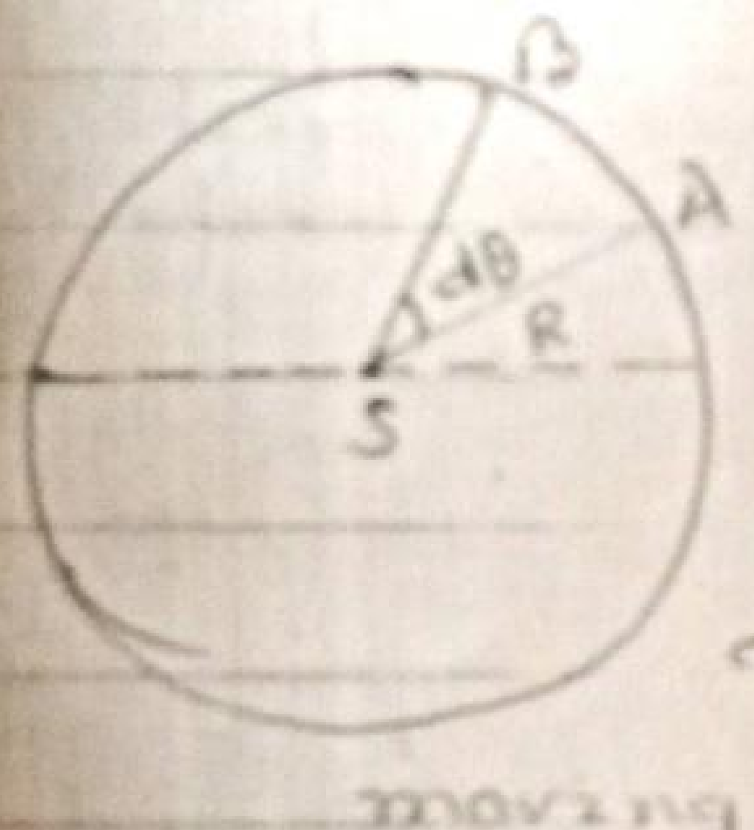
Ques:- State and Explain Kepler's law of planetary motion. Derive Newton's law of motion from Kepler's law of motion.

Ans:- The three laws of planetary motion are

i) Shape of the Orbit :- Every planet moves in an elliptical orbit with the Sun being at one of its foci.

ii) Velocity in the orbit [law of Area] :- The radius vector drawn from the sun to the planet sweeps out equal areas in equal intervals of time.



Consider a planet moving in an elliptical orbit with the sun at the focus S. Let the radius vector of the planet describes a small angle dθ in small interval of time dt while moving from the position A to B.

$$\text{Arc } AB = R\,d\theta$$

$$\text{Area swept in time } dt = \text{area } ABS = \tfrac{1}{2} \times R \times R\,d\theta = \tfrac{1}{2} R^2\,d\theta$$

$$\text{Areal velocity} = \tfrac{1}{2} R^2\,d\theta/dt = \tfrac{1}{2} R^2 \left(\frac{d\theta}{dt}\right) = \tfrac{1}{2} R^2 \omega = \text{constant}$$

or, Areal velocity = a constant.

As areal velocity $\frac{1}{2}R^2\omega$ is constant, therefore, ~~$\frac{1}{2}R^2\omega$~~ $\frac{1}{2}MR^2\omega$ is also constant, where M is the mass of the planet. Therefore $MR^2\omega$ remains constant throughout the motion of a planet in its orbit. The term $MR^2\omega$ represents the angular momentum of the planet. Hence the laws of areas is a statement of the law of conservation of angular momentum.

$$\text{Area of the ellipse} = \pi ab$$

where a and b are semi-major and semi-minor axes of the ellipse respectively.

Time period of revolution of the planet around the sun.

$$T = \frac{\text{Area}}{\text{Areal velocity}} = \frac{\pi ab}{\frac{1}{2}R^2\omega}$$

$$\text{or, } T = 2\pi ab / R^2\omega$$

iii) Time period of planets:- The square of the time period of revolution of a planet around the sun is proportional to the cube of the semi-major axis of the orbit.

Thus if T_1 and T_2 are the semi major axis of the planets respectively

$$\frac{T_1^2}{a_1^3} = \frac{T_2^2}{a_2^3} = \text{constant}$$

$$\text{or, } \frac{T^2}{a^3} = \text{Constant}$$

$$T^2 \propto a^3$$

Derivation of law of Gravitation :-

Suppose the mass of a planet A is M_1, the radius of its orbit is R_1 and time period of revolution is T_1. It is assumed that the orbit is circular. The force of attraction exerted by the sun on the planet (centripetal force).

$$F_1 = M_1 R_1 \omega^2$$

$$\text{or, } F_1 = M_1 R_1 \left(\frac{2\pi}{T_1}\right)^2 \quad \text{———— (1)}$$

Similarly for a second planet B of mass M_2 radius R_2 and period of revolution round the sun is T_2, then

$$F_2 = M_2 R_2 \left(\frac{2\pi}{T_2}\right)^2 \quad \text{———— (2)}$$

Hence $\frac{F_1}{F_2} = \left(\frac{M_1}{M_2}\right)\left(\frac{R_1}{R_2}\right)\left(\frac{T_2}{T_1}\right)^2$ ——— (3)

But according to the kepler's third law,

$$\left(\frac{T_2}{T_1}\right)^2 = \left(\frac{R_2}{R_1}\right)^3$$

Substituting these values in eqn (3) we get

$$\frac{F_1}{F_2} = \frac{M_1}{M_2}\left(\frac{R_2}{R_1}\right)^2$$

or, $\frac{F_1 R_1^2}{M_1} = \frac{F_2 R_2^2}{M_2}$

or, $\frac{F R^2}{M} = \text{Constant}$

or, $F \propto \frac{M}{R^2}$

or, (i) $F \propto M$ and (ii) $F \propto \frac{1}{R^2}$

Thus the force of attraction exerted by the sun on the a planet is proportional to its mass and inversely proportional to the square of its distance from the sun.

Thus we can derive Newton's law of gravitation from kepler's law of planetary motion. Here the constant of proportionality is equal to GM' where G is the universal gravitational constant and M' is the mass of the sun which is itself constant.

We can write it in mathematical expression as

(i) $F \propto \frac{M}{R^2}$ or, $F = \frac{GM'M}{R^2}$

This is Newton's law of gravitation.

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